SOME ASPECTS OF PROBLEMS IN MAGNETOHYDRODYNAMIC INSTABILITY THEORY WHICH CAN BE REDUCED TO A DIFFERENTIAL EQUATION WITH AN ARBITRARY PARAMETER ASSOCIATED WITH THE LEADING DERIVATIVE

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The asymptotic properties of the solutions of a fourth-order differential equation with an arbitrary parameter associated with the leading derivative are investigated. It is shown that the similarity of the asymptotic forms is independent of the value of the parameter associated with the leading derivative when the coefficient of the second derivative has zeros.
The influence of longitudinal current, a finite ion Larmor radius and ion-ion viscosity on the stability of a plasma in a magnetic field are considered, and the stabilizing effect of ion rotation in the magnetic field is also determined.

1. It is well known that in the investigation of the oscillation spectra in ordinary hydrodynamics and in magnetohydrodynamics the problem reduces, in a number of cases, to a differential equation of higher than second order [1, 2]. In particular, the case when the coefficient of the (n − 2) derivative in an n-th order differential equation vanishes at some point in the complex X plane plays a special part. In this connection an investigation of the differential equation

$$\alpha\beta^2\varphi^{\mathrm{IV}} - \beta u_2(x)\varphi'' + u_1(x)\varphi = 0 \qquad (1.1)$$

was undertaken in [3, 4] with two small parameters: β the "quasi-classical" small parameter characterizing a weakly irregular medium, α another small parameter (associated, for example, with the influence of weak viscosity); we note that α and β may, in particular, be equal to unity. The coefficients u_2, $u_1 \sim 1$, except at those points where they become zero.

In [3, 4] particular attention was paid to the influence of "intersection" points in the solutions on the nature of the oscillation spectra. In particular, close to the point $u_2 = 0$, there are two "intersection" points of the solutions, and the distance between them, proportional to the parameter α/β^2, determines the dimension of the singular region in the case in question. The methods of solution have been classified according to the parameter α/β^2, particular attention being paid to the case $\alpha/\beta^2 \gg 1$, when the method of phase integrals may be employed in order to pass around each singularity separately, since the distance between points where the solutions intersect is large compared with the characteristic wavelength of the intersecting solutions.

In the present paper it will be shown that by using Laplace's method in association with the analytic properties of the solutions it is possible, from a single viewpoint, to investigate the problem for any values of the parameter α/β^2. It was noted in [3] that in accordance with the results of [5] we must expect an anomalous transition from one solution to another in some region of the complex X plane close to the point $u_2 = 0$, and also for $(\alpha/\beta^2) \ll 1$. As will be clear from what follows, this is associated with the fact that the solutions possess, in a known sense, identical asymptotic properties for arbitrary values of α/β^2. For simplicity, we confine ourselves to fourth-order differential equations.

We shall consider the differential equation

$$\varphi^{\mathrm{IV}} + \lambda^2(x\varphi'' + u_{10}\varphi) = 0 \qquad (1.2)$$

which was investigated in [6] for large values of the parameter λ. It was shown in [3] that (1.1) may be reduced to a similar form close to the point $u_2 = 0$, while $\lambda^2 = \beta^2/\alpha$.

Using Laplace's method, we write the solution of Eq. (1.2) as

$$\varphi(x) = \int t^{-2}\exp\left(tx - \frac{u_{10}}{t} + \frac{t^3}{3}\lambda^{-2}\right)dt. \qquad (1.3)$$

Just as in [6], we choose the contour C as shown in the figure. The ends of the contours, which go off to infinity, lie in sectors of the t plane where $\mathrm{Re}\,(t^3/\lambda^2) < 0$. Then, in accordance with Cauchy's theorem, we have the following relation between the solutions:

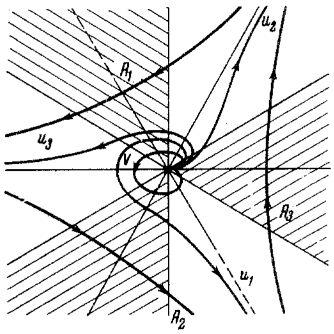
$$A_1 + A_2 + A_3 = V, \quad u_3 - u_2 = A_1, \quad u_1 - u_3 = A_2. \qquad (1.4)$$

We shall determine the behavior of the solutions indicated above on the real axis X for small values of the parameter λ. First of all, we shall consider the solution V. On calculating the residue for t = 0, we obtain

$$V = 2\pi i\left(\frac{x}{u_{10}}\right)^{1/2}\left\{J_1(2\sqrt{xu_{10}}) + \sum_{n=1}^{\infty}\frac{J_{3n-1}(2\sqrt{xu_{10}})}{n!\,a^n}\right\},$$

$$\left(a = -3\lambda^2\left(\frac{x}{u_{10}}\right)^{3/2}\right). \qquad (1.5)$$

The summation in expression (1.5) may be neglected at sufficiently large values of x, when $|a| \gg 1$, and the expression passes into the corresponding expression obtained in [6].



We shall now consider the solutions A_k. Making the change of variable $t = \lambda x^{1/2}\sigma$, we have

$$A_k = \frac{1}{\lambda\sqrt{x}}\int_{c_k}\exp\left[\lambda x^{3/2}\left(\sigma + \frac{\sigma^3}{3}\right) - \frac{u_{10}}{\sigma\lambda x^{1/2}}\right]\frac{d\sigma}{\sigma^2}. \qquad (1.6)$$

We see from (1.6) that in the case where $\lambda \gg 1$, the term $u_{10}\sigma^{-1} / \lambda x^{1/2}$ of the exponential may be neglected, so that further calculations are quite similar to those carried out in [6]. Similar arguments may be brought forward as regards the solutions U_k. We now give the asymptotic expression for A_3 on the real axis

$$A_3 \approx \sqrt{\pi}\lambda^{-2/3}e^{-1/3 i\pi}\left(\frac{3\xi}{2}\right)^{-5/6} e^{\xi}, \qquad \arg x = -\pi,$$

$$A_3 \approx \sqrt{\pi}\lambda^{-2/3}e^{1/4 i\pi}\left(\frac{3\xi}{2}\right)^{-5/6} [e^{\xi} + ie^{-\xi}] + V,$$

$$\arg x = 0, \;\; \xi = i\lambda x^{3/2}.$$

We note that the expressions given here coincide with those obtained by the method of phase integrals. In the sector $\arg x = \pi + {}^1/_3\gamma\pi$, $|\gamma| < 1$ the solution A_3 behaves as the most rapidly descending solution and does not contain "admixtures" of other solutions. In the sector $\arg x = {}^2/_3\gamma\pi$, within the limits of error of the saddle point method, the solution V has no effect on the behavior of A_3.

Finally, on the line $x > 0$ both types of solution are purely oscillatory and have a "weak" difference of orders, expressed in powers of x and λ. One easily notes that close to the lines c_k, satisfying the condition $\arg x = {}^2/_3\pi\,(k - 2)$, terms having the form of solutions of type V attach themselves to the solutions A_k (which are purely oscillatory on these lines). This is associated with the disposition of the saddle points and the lines of descent relative to the contours of the solutions, which determines what possibilities we have of deforming the contour in the sector of the lines of descent. Close to the lines c_k the saddle points $t_{1,2}{}^2 = -\lambda^2 x$ fall on the border of the unshaded regions, and the contour $C(A_k)$ is deformed into lines passing through all the saddle points. We note that for solutions of the type U_k, in view of their lesser degree of growth, there exists a sector in the x plane in which there is an admixture of increasing (or purely oscillatory) A_k, since it is impossible to eliminate the influence of the saddle points $t_{1,2}$ by deforming the contour. Since the asymptotic forms are similar for $\lambda \ll 1$ and $\lambda \gg 1$, in both the cases indicated the rules for finding the frequency spectrum for finite solutions are similar to those obtained in [3].

Of course, the boundary conditions on the walls may lead to other rules for locating the frequencies and, in particular, may violate the strong link between the solutions.

We note that in the case under consideration both $k_1 - k_2$ and $k_1 + k_2$ experience branching in the neighborhood of the singular region $u_2 = 0$. Here $k_1(x)$, $k_2(x)$ the wave vectors of the branching solutions are easily determined from (1.1). In the case where only $k_1 - k_2$, for example, experiences branching, and the branching points are not situated on the real axis, the relation between the solutions becomes weaker [7].

2. Before passing to concrete applications of the theory, we shall make some further observations. Firstly, it follows from the foregoing discussion that there is no necessity to investigate the size of the "resonance" region between the "intersection" points of the solutions, since the oscillation spectrum of finite solutions does not depend on this when the point $u_2 = 0$ is present. All that is required is that the width of the "resonance" region should be small compared with the width of the "hole" in which the finite solution is localized; however the width of the "hole" is equal to $\sim R$ (R is the characteristic dimension of an inhomogeneity), and the dimension of the "resonance" region is $\sim R\sqrt{\alpha}$. Thus the latter requirement is in fact always fulfilled. Moreover, since in concrete problems the coefficient u_2 is, generally speaking, a function of frequency, the disposition of the points $u_2 = 0$ in the complex X plane plays an important part. If the points $u_2 = 0$ are situated remote from the real axis (this is usually the case if $\mathrm{Re}\,\omega \sim \mathrm{Im}\,\omega$), and finite solutions of (1.1) exist on the real axis, corresponding to a given frequency, then the points $u_2 = 0$ exert a weak influence on such solutions. Putting it another way, the theory considered here is, generally speaking, applicable to the case

$$\mathrm{Im}\,\omega \ll \mathrm{Re}\,\omega\,.$$

We also note that singularities of the type of "intersection" of solutions or "reversal points" will exert an influence on a disturbance in the form of a wave-packet if

$$\frac{d(\mathrm{Re}\,\omega)}{dk}\,\frac{1}{\mathrm{Im}\,\omega}\ln\frac{A}{A_m} > L\,.$$

Here L is the distance from the place where the packet is localized to the region where the geometrical-optics approximation is violated, A_m is the amplitude of the initial "noise" in the medium, A is the amplitude of the disturbance in the nonlinear mode. Otherwise it is not necessary to allow for nonlinear effects.

We shall now consider the effect of a finite ion Larmor radius, and also of ion-ion viscosity on the development of plasma instabilities in a magnetic field as a result of the longitudinal current [8, 9]. The equation for the perturbed quantities has the following form [9, 10]:

$$\alpha\varphi^{\mathrm{IV}} + k_y{}^2\left[-2\alpha + \frac{\omega - \omega_i}{i\nu_i}\right]\varphi'' + k_y{}^4\left[\alpha - \frac{\omega - \omega_i}{i\nu_i} + \right.$$
$$\left. + \frac{\omega - \omega_i}{\nu_i}\,\frac{\omega_s}{\omega}\left(1 - \frac{\omega_e}{\omega - \omega_i} + \frac{i\omega_0}{\omega - {}^2/_3\, i\chi k_z{}^2 - \omega_i}\right)\right]\varphi = 0. \quad (2.1)$$

Here

$$\alpha = (k_y r_i)^2\,, \qquad \omega_e = k_y\frac{cT_{0e}}{eH_0}\,\frac{1}{n_0}\,\frac{dn_0}{dx}\,,$$

$$\omega_i = k_y\frac{cT_{0i}}{eH_0}\,\frac{1}{n_0}\,\frac{dn_0}{dx}\,, \qquad \omega_s = \left(\frac{k_z}{k_y}\right)^2\frac{\omega_{Hi}\omega_{He}}{\nu_e}\,,$$

$$\chi = \frac{\varkappa}{n_0}\,, \qquad \omega_0 = \frac{k_y}{k_z}\,\frac{j_0 c}{H_0\sigma_{\parallel}{}^2}\,\frac{d\sigma_{\parallel}}{dx}\,. \qquad (2.2)$$

The following notation has been introduced in (2.1) and (2.2): k_y, k_z the components of the wave vector on the y and z axes, respectively; r_i the ion Larmor radius; H_0 the magnetic field directed along the z axis; T_{0e}, T_{0i}, ω_{He}, ω_{Hi} the electron and ion temperatures and Larmor frequencies, respectively; j_0 the initial longitudinal current; $\sigma_\parallel$ the longitudinal conductivity; ν_e, ν_i the collision frequencies of electrons with ions and ions with ions, respectively; n_0 the unperturbed density; e the electronic charge; c the velocity of light; $\varkappa$ the electronic thermal conductivity. It is convenient to carry out qualitative investigations of stability with the help of the relation [10]

$$\operatorname{Im}(u_1 + \beta u_2 |k|^2) = 0,$$
$$\operatorname{Re}(\alpha\beta^2 k_x^4) - \operatorname{Re}(u_1 + \beta u_2 k_x^2) = 0, \qquad (2.3)$$

written for some point $x^{(p)}$ in the region of localization of $\varphi(x)$ for (1.1). Instead of k_x in (2.3) either k_1 or k_2 is taken, depending on the summed contribution of modes in the stability criterion $k_1 / k_2 \sim \sqrt{\alpha}$. First of all, we shall determine how a finite ion Larmor radius influences the instability due to the longitudinal current if the coefficient of the second derivative does not vanish. In this case we need not take into account the fourth derivative in Eq. (2.3). Considering for simplicity the case when $\omega_0 \gg \omega_s, \omega_e, \alpha\nu_i, \chi k_z^2$, we obtain from (2.3)

$$\omega_{1,2} = {}^1/_2\omega_i \pm {}^1/_2\sqrt{\omega_i^2 - 4\omega_0\omega_s}\,. \qquad (2.4)$$

We see from (2.4) that it is possible to stabilize the current instability due to the finite size of the ion Larmor radius if

$$\omega_i^2 > 4\omega_0\omega_s.$$

Now if there is a point $u_2 = 0$ in the neighborhood of the real axis, then the influence of k_2 "modes" will be substantial. We then obtain the following stabilization condition from (2.3)

$$\omega_i^2 > 4\alpha\omega_0\omega_s,$$

i.e., the stabilization condition is notably improved.

We shall now calculate the effect of ion-ion viscosity which corresponds to the term $\alpha\nu_i$. Then on condition that $\omega^2 \gg 4\omega_0\omega_s$ (from which, in particular, there follows the stabilization of the usual current instability), and also that $\omega_i \gg \alpha\nu_i$, we have from (2.3), when the fourth derivative may be neglected,

$$\omega = \frac{\omega_0\omega_s}{\omega_i} - i\alpha\nu_i\frac{\omega_0\omega_s}{\omega_i^2}\,. \qquad (2.5)$$

We see from (2.5) that the simultaneous taking into account of the longitudinal current, the finite size of the ion Larmor radius and the ion-ion viscosity leads once again to an instability with a small increment ($\operatorname{Im}\omega \ll \sqrt{\omega_0\omega_s}$). When points $u_2 = 0$ are present it follows from (2.3) that Re ω and Im ω are diminished by a factor of α, but the instability remains.

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